Highly arc-transitive digraphs – counterexamples and structure*

Matt DeVos † Bojan Mohar†‡§ Robert Šámal ¶ ‖

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Abstract

We resolve two problems of [Cameron, Praeger, and Wormald – Infinite highly arc transitive digraphs and universal covering digraphs, Combinatorica 1993]. First, we construct a locally finite highly arc-transitive digraph with universal reachability relation. Second, we provide constructions of 2-ended highly arc transitive digraphs where each 'building block' is a finite bipartite graph that is not a disjoint union of complete bipartite graphs. This was conjectured impossible in the above paper. We also describe the structure of 2-ended highly arc transitive digraphs in more generality, although complete characterization remains elusive.

1 Introduction

A digraph D consists of a set of vertices $V(D)$ and edges/arcs $E \subseteq V(D) \times V(D)$; in this paper we consider digraphs without loops. An *s-arc* is an $(s+1)$-tuple of vertices $(v_0, v_1, \dots, v_s)$ such that $v_{i-1}v_i$ is an edge for each $i = 1, \dots, s$ and $v_{i+1} \neq v_{i-1}$ for each $i = 1, \dots, s-1$. A digraph D is *s-arc transitive* if for every two s-arcs $(v_i)_{i=0}^s$, $(v'_i)_{i=0}^s$ there is an automorphism f of D such that $f(v_i) = v'_i$

*Emails: {mdevos,mohar}@sfu.ca, samal@kam.mff.cuni.cz

†Dept. of Mathematics, Simon Fraser University, Burnaby, Canada

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§On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

¶Dept. of applied math., Faculty of Math. and Physics, Charles University, Prague, Czech Republic.

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for each i. If the digraph is symmetric (that is, each edge is in a 2-cycle) then an s-arc corresponds to a nonretracting path of length s in the underlying undirected graph. Celebrated result of Tutte [7] states that a finite 3-regular graph can be s-arc transitive only if $s \leq 5$. Weiss [8] extended this (using the classification of finite simple groups) to finite r-regular graphs ($r > 2$); these can be s-arc transitive only if $s \leq 8$. (Somewhat trivially, cycles are s-arc transitive for every s.)

A digraph is *highly arc transitive* if it is s-arc transitive for every s. As one may expect, this is very demanding definition. Indeed, the only *finite* highly arc transitive digraphs are the directed cycles. Among infinite digraphs, the number of highly arc transitive ones is much larger. Still, they are rather restricted, which makes the constructions nontrivial, and one may hope to characterize all such digraphs, at least to some extent.

An obvious infinite highly arc transitive digraph is the doubly infinite directed path, which we shall denote by Z. Another immediate example is obtained when we replace each vertex of Z by an independent set of size k and every arc by a (directed) complete bipartite graph $\vec{K}_{k,k}$ – formally this is a *tensor product* $Z \otimes \bar{K}_k$ with $\bar{K}_k$ denoting the complete digraph with k vertices and all k^2 edges (here including the loops). Confirm also Lemma 4.3 and Theorem 4.5 of [1] for more on tensor product and high arc-transitivity.

The question of what other highly arc transitive digraphs exist has started a substantial amount of research. The question was originally considered by Cameron, Praeger, and Wormald [1]. They presented some nontrivial constructions (details can be found in Section 3) and worked on ways to describe all highly arc transitive digraphs. One approach to this involves the *reachability relation.*

Given a digraph D, an *alternating walk* is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_i v_{i+1}$ and $v_i v_{i-1}$ are arcs of D *either for all even i or for all odd i*; informally, when visiting the vertices $v_0, \ldots, v_s$, we use the arc of D alternately in the forward and backward directions. When e, e' are two arcs of D, we say that e' is *reachable* from e if there is an alternating walk which has e as the first arc and e' as the last one (we write $e \sim e'$). One can easily see that this is an equivalence relation, moreover one that is preserved by any digraph automorphism. Thus, whenever D is 1-arc transitive, then the digraph induced by each of the equivalence relations is isomorphic to one fixed digraph, that we call $R(D)$ (R stands for reachability, previous papers use $\Delta(D)$, which seems unfortunate, as Δ usually denotes the maximum degree of a graph).

It is shown in [1] that if the reachability relation has more than one class, then $R(\Delta)$ is bipartite and a construction is presented, that for arbitrary directed bipartite digraph R gives a highly arc transitive digraph D with $R(D) \simeq R$; in fact a universal cover for all such D's is constructed. Thus a question arises, whether there are highly arc transitive digraphs for which the reachability relation is *universal* (by which is meant that is has just one equivalence class); as this approach to classify highly arc transitive digraphs would not work for them.

Actually, such digraphs are rather easy to construct if we allow infinite degree, one example would be Q: $V(Q) = \mathbb{Q}$, $uv \in E(Q)$ iff $u < v$. So, the following question was asked in [1]:

Question 1.1 *Is there a locally finite (i.e., all degrees finite) highly arc transitive digraph with universal reachability relation?*

In Section 2 we present a construction of such digraph — showing, in effect, that highly arc transitive digraphs form a richer class of graphs than one might expect.

Many highly arc transitive digraphs posses a homomorphism onto Z – that is, a mapping $f : V \to \mathbb{Z}$ such that for every edge uv we have $f(v) = f(u) + 1$. This is called *property* Z in [1], and the authors ask, whether all locally finite highly arc transitive digraphs have this property. A counterexample was given by [4]. Another counterexample is provided by our graph with a universal reachability relation, as a graph with property Z has infinitely many equivalence classes.

Another approach to classify highly arc transitive digraphs is to use the number of *ends*. (See [2] for the definition of an end of a graph.)

It is easy to see that a highly arc transitive digraph may have 1, 2, or infinitely many ends. An example with two ends is Z, with infinitely many ends a tree (where the in-degree of all vertices is some constant d^- and the out-degree of all vertices is some constant d^+). An example of highly arc transitive digraphs with just one end is Q. A locally finite example is harder to construct, see [5].

Let us focus on two-ended digraphs. This includes the basic examples Z, $Z \otimes \bar{K}_k$ already mentioned, as well as a more complicated construction by McKay and Praeger discussed in Section 3. This construction was generalized in [1]. The authors also conjectured that for each connected highly arc transitive digraph D the graph $R(D)$ is either infinite, or a complete bipartite graph. We disprove this conjecture – in Section 3 we present several constructions that behave in a more complicated way. Finally, in Section 4 we work towards characterizing all two-ended highly arc transitive digraphs. While we provide some answers, there is still many gaps to be filled by either providing more constructions or a better characterization.

2 The highly arc transitive digraph with universal reachability relation

The following answers Question 1.2 of [1].

Theorem 2.1 *There is a locally finite highly arc-transitive digraph for which the reachability relation is universal.*

In fact, for every non-prime integer $d \geq 4$ there is such digraph with all in-degrees and all out-degrees equal to d.

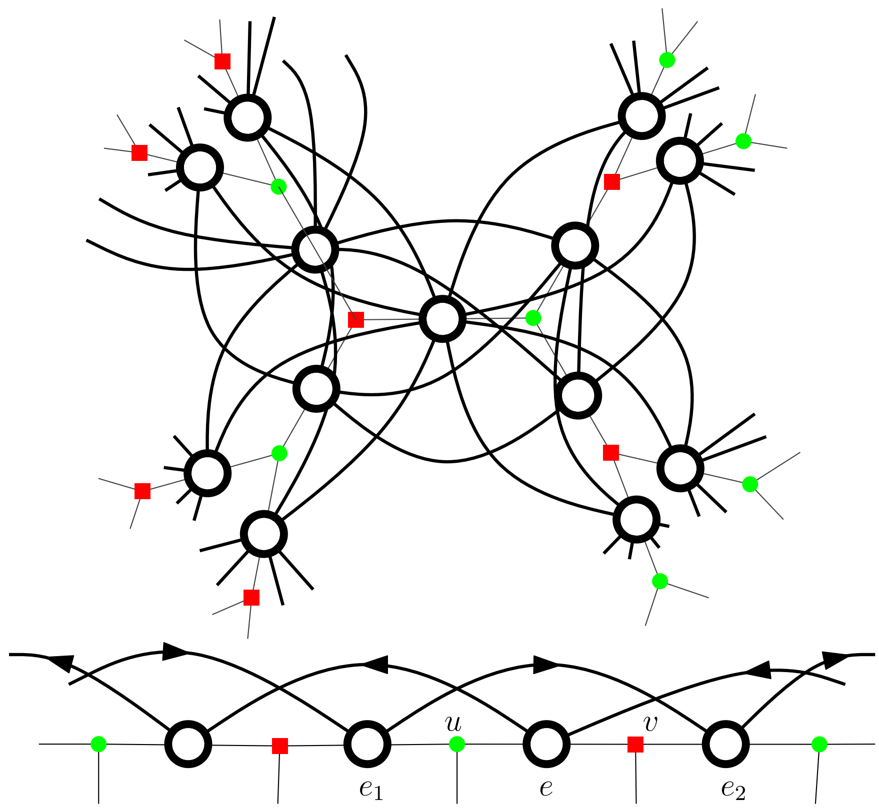

Figure 1: The graph $G_{3,3}$. Above, a part of the graph (with the underlying tree), without directions of edges. Below, description of the direction of edges. Vertices of the set A are circles, vertices of B squares.

Proof: Pick integers $a, b \geq 3$. We will construct such digraph where every vertex has in- and out-degree equal to $(a-1)(b-1)$. Let $T = T_{a,b}$ be the infinite tree with vertex set $A \dot{\cup} B$, where every vertex in A has a neighbours in B, and every vertex in B has b neighbours in A. Next, we define the desired digraph with $V(G_{a,b}) = E(T_{a,b})$. For each $e = uv \in E(T_{a,b})$, where $u \in A$, $v \in B$, we add an arc from each e_1 adjacent to e at u to each e_2 adjacent to e at v (an edge is not considered adjacent to itself). For each such e_1, e_2 we put $c(e_1, e_2) := e$. We let $G = G_{a,b}$ be the resulting digraph; in Fig. 1 we display part of $G_{3,3}$.

First we prove that G is highly arc-transitive. Suppose $\mathbf{e} = (e_0, e_1, \ldots, e_s)$ is an s-arc in G, and let $P(\mathbf{e})$ be $e_0, c(e_0, e_1), e_1, \ldots, c(e_{s-1}, e_s), e_s$, a path in T. Now let $\mathbf{e}'$ be another s-arc in G. Obviously $P(\mathbf{e})$ and $P(\mathbf{e}')$ are paths in T of the same length, both starting at a vertex of B. Consequently, there is an automorphism φ of T that maps $P(\mathbf{e})$ to $P(\mathbf{e}')$. The mapping that φ induces on $E(T) = V(G)$ is clearly an automorphism of G that sends $\mathbf{e}$ to $\mathbf{e}'$.

We still need to show that the reachability relation of G is universal. Suppose $e, e' \in V(G)$ are adjacent as edges in T, and that a (resp. a') is an arc of G starting at e (resp. e'). We will show that $a \sim a'$, this is clearly sufficient. Assume first that e and e' share a vertex of A. Let a_1, a_2 be arcs of G as in Fig. 2 on the left (recall that $a \geq 3$). Obviously a, a_1, a_2, a' is an alternating walk, thus $a \sim a'$. Secondly, assume e and e' share a vertex of B. In this case pick arcs a_1, a_2 according to Fig. 2 on the right, utilizing that $b \geq 3$. Now $a \sim a_1$ and $a_2 \sim a'$ according to the first case and, clearly, $a_1 \sim a_2$. This finishes the proof. $\square$

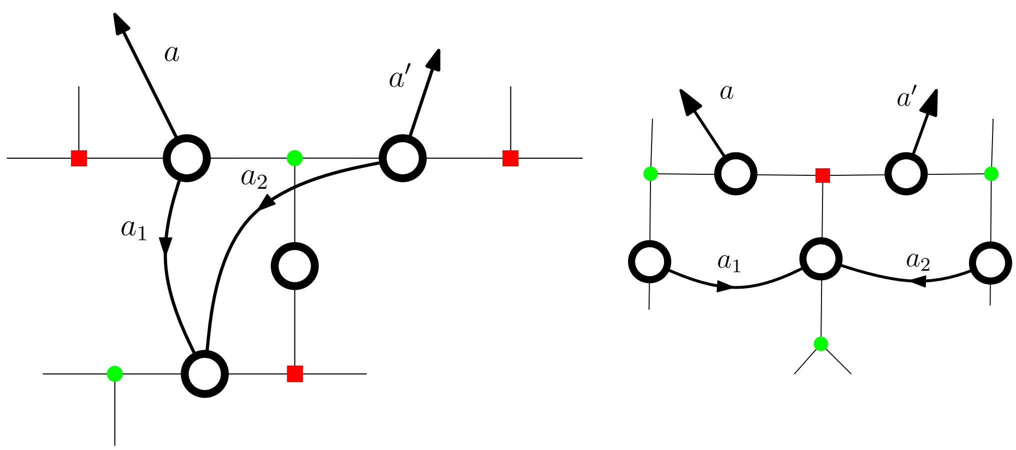

Figure 2: Two arcs of $G_{a,b}$ that start at adjacent edges of $T_{a,b}$ are equivalent.

Remark: It is known that highly arc-transitive digraph with universal reachability relation does not exist if $d^+ \neq d^-$ [6] neither if $d^+ = d^-$ is a prime [1]. Whenever $d^+ = d^-$ is not a prime, it can be written as $(a-1)(b-1)$ for $a, b \geq 3$, so our construction provides an example of such digraph.

Note that the structure of the graph $G_{a,b}$ can also be described as follows. Consider a partitioning of $K_{a,a(b-1)}$ into a copies of a star, $K_{1,b-1}$; we call these copies $S_1, \ldots, S_b$. We let H be $K_{a,a(b-1)} - \cup_i E(S_i)$. Then we take countably many copies of H and glue them (in a "tree-like" fashion) on the sets corresponding to some S_i. From this description it is immediate that $G_{a,b}$ has universal reachability relation.

3 Two-ended constructions

As mentioned in the introduction, a highly arc transitive digraph can have 1, 2, or infinitely many ends; in the rest of this paper we concentrate on the case of two ends. It is not hard to show (see the proof of Proposition 4.1 that every 1-arc transitive digraph D has the following structure: the vertices can be partitioned as $V(D) = \bigcup_{i=-\infty}^{\infty} V_i$ and all arcs go from some V_i to V_{i+1}. Moreover, if D is

also 0-arc- (i.e., vertex-) transitive then each of the induced graphs $D[V_i \cup V_{i+1}]$ is isomorphic to a fixed bipartite 'tile' B. If B is a complete bipartite digraph $\vec{K}_{k,k}$, we get again the basic example $Z \otimes \bar{K}_k$. For other B's, the digraph D is not determined just by B, as we need to specify how are the copies of B 'glued'. It is easy to see that all components of B are isomorphic to $R(D)$. The following was conjectured in [1].

Conjecture 3.1 ([1]) *If D is a connected highly arc transitive digraph such that a homomorphism $f : D \to Z$ exists and $f^{-1}(0)$ is finite, then $R(D)$ is a complete bipartite graph.*

Next, we describe several constructions. We start with the one by Praeger and Wormald (Remark 3.4 of [1]), that, while nontrivial, confirms the above conjecture. Then we present our construction, disproving the conjecture, and then some generalizations that, as we hope, are not too far from the general characterization.

Construction 1 (McKay, Praeger [1]) We let $V = \mathbb{Z} \times S^n$ (for a finite set S). Vertices $a = (i, a_1, \dots, a_n)$ and $b = (i+1, b_1, \dots, b_n)$ are connected iff $a_j = b_{j+1}$ for each $j = 1, \dots, n-1$, no other edges are present.

Here the graph B is a disjoint union of complete bipartite graphs (more precisely, $|S|^{n-1}$ copies of $\vec{K}_{|S|,|S|}$), thus $R(D)$ is $\vec{K}_{|S|,|S|}$. The fact, that this is a highly arc transitive digraph is shown in [1], but also follows from our next constructions.

Construction 2 Let T be a "template" – an arc transitive digraph that is bipartite with parts A_1, A_2, all arcs directed from A_1 to A_2. Put $V = \mathbb{Z} \times A_1 \times A_2$. Vertices (i, a_1, a_2), $(i+1, b_1, b_2)$ are connected iff $(a_1, b_2) \in E(T)$, no other edges are present. We call the resulting digraph D, also we denote $V_i = \{a \in V(D) : a_0 = i\}$.

The bipartite tile B is obtained from T by taking $|A_2|$ copies of each vertex in A_1 and $|A_1|$ copies of each vertex in A_2. If T is connected, then B is isomorphic to $R(D)$. Thus, for T being $\vec{K}_{3,3}$ minus a matching (alternately oriented 6-cycle) we get a counterexample to Conjecture 3.1.

Construction 3 The next construction generalizes at the same time Construction 1 and Construction 2. Let T be a $(t-1)$-arc-transitive template digraph, with vertices in t "levels", A_1, ..., A_t and all arcs lead from A_j to A_{j+1} for some j. Suppose that each vertex v of T has in-degree at least 1 (unless $v \in A_1$) and out-degree at least 1 (unless $v \in A_t$).

Put $V = \mathbb{Z} \times A_1 \times A_2 \times \cdots \times A_t$.

Vertices $a = (i, a_1, a_2, \dots, a_t)$, $b = (i+1, b_1, b_2, \dots, b_t)$ are connected iff $(a_j, b_{j+1}) \in E(T)$ for each $j = 1, \dots, t-1$, no other edges are present. We

denote the resulting graph by $D = D(T)$. Clearly, for $t = 2$ we get Construction 2. Construction 1 of McKay and Praeger is a special case of this one, with D consisting of $|S|$ disjoint paths.

Theorem 3.2 *If T is as above, then the constructed digraph $D(T)$ is highly arc transitive.*

Proof: First we show that $D(T)$ is connected. Easily, the following stament suffices for this: for every $a \in V_0$, $b \in V_t$ there is a directed a–b path. In order to prove this, observe that every vertex of T is a part of at least one directed path with t vertices. Let P_i (Q_i, resp.) be such path containing a_i (b_i, resp.). We let $P_{i,j}$ denote the j-th vertex on the P_i, so that $P_{i,i} = a_i$ (and, similarly, $Q_{i,i} = b_i$). Now we define vertices forming a directed path $c_0 = a, c_1, \ldots, c_s = b$. We put

$$c_{i,j} = \begin{cases} P_{j-i,j} & \text{if } i < j, \\ Q_{j+s-i,j} & \text{if } i \geq j. \end{cases}$$

Next, we describe some automorphisms of $D(T)$. A trivial one is $\tau : a \mapsto (a_0 + 1, a_1, \ldots, a_t)$ – a shift in the first coordinate. More interesting automorphisms come from automorphisms of T: Let φ be an automorphism of T. By φ_k we denote a mapping that applies φ on the j-th coordinate in V_{k+j} whenever $1 \leq j \leq s$. We shall show that φ_k is an automorphism of $D(T)$: suppose $ab \in E(D)$ and φ_k affects j-th coordinated in a (precisely: $\varphi_k(a)_j \neq a_j$). Now either $j = t$ or $j < t$. In the former case, $\varphi(b) = b$, and the coordinate a_j is not tested in the definition of edges of $E(D(T))$, so $\varphi(a)\varphi(b)$ is an edge, too. In the latter case φ_k changes $(j+1)$-st coordinate of b, and

$$\varphi_k(a)_j = \varphi(a_j)$$
$$\varphi_k(b)_{j+1} = \varphi(b_{j+1})\,.$$

By assumption ab is an edge of $D(T)$, so $a_j b_{j+1}$ is an edge of T, and as φ is an automorphism of T, $\varphi(a_j)\varphi(b_{j+1})$ is an edge as well.

We show now that the group generated by

$$\{\tau\} \cup \{\varphi_k : \varphi \in \mathrm{Aut}(T), k \in \mathbb{Z}\}$$

acts transitively on s-arcs. Let $(v_i)_{i=0}^s$, $(v'_i)_{i=0}^s$ be two s-arcs in $D(T)$. By applying τ or τ^{-1} we may assume that $(v_0)_0 = (v'_0)_0 = 0$ (thus also $(v_i)_0 = (v'_i)_0 = i$ for each i). If $v_i = v'_i$ for each i then we are done, otherwise find i and j so that $(v_i)_j \neq (v'_i)_j$ and $k = i - j$ is minimal. (*)

We put $a_\ell = (v_{\ell+j})_\ell$ for $\ell = 1, \ldots, t$; if $\ell + j < 0$ or $\ell + j > s$ then we pick a_ℓ arbitrarily. Similarly, we define a'_ℓ from v'. Now (a_ℓ), (a'_ℓ) are two $(t-1)$-arcs

in T, thus there is an automorphism φ of T such that $\varphi(a_\ell) = a'_\ell$. Observe that s-arcs $(\varphi_k(v_i))$ and (v'_i) are closer (so that we shall get larger k in (*)) than (v_i) and (v'_i), so after repeating this procedure at most $s+t$ times we map one s-arc to the other. □

As the requirements on the template T are rather strong, we describe next a nice source of nontrivial templates. Consider a finite space (affine – $AG(n,q)$ or projective – $PG(n,q)$), let A_i denote its subspaces of dimension $i-1$. We let the arcs denote incidence – (x,y) is an arc iff x is a subspace of y and $\dim y = \dim x + 1$. This gives a template with $t = n-1$. A $(t-1)$-arc corresponds to a flag (that is, a sequence of a subspaces one contained in another, one of each dimension). It is not hard to show that the geometric space is flag-transitive, thus the resulting template is $(t-1)$-arc transitive.

A natural question remains: do we get more highly arc transitive digraphs by Construction 3, that were not obtainable by Construction 2? The answer is positive, to prove it, let us first define a notion of *clones*. Given a digraph, we call vertices x, x' *right clones*, if they have the same outneighbours (xy is an edge iff $x'y$ is an edge); we call them *left clones* if they have the same inneighbours. It is not hard to show that in a highly arc transitive digraph, all vertices have the same number c^+ of right clones and the same number c^- of left-clones. In Construction 2 we have $c^+ \geq |A_2|$ and $c^- \geq |A_1|$, so $c^+c^- \geq |V_0|$. On the other hand, using Construction 3 with a template T from finite geometries we have $c^+ = |A_t|$ and $c^- = |A_1|$ so for $t > 2$ we have $c^+c^- < |V_0|$, hence we get a new highly arc transitive digraph.

4 Structure in the two-ended case

The goal of this section is to prove a structural result concerning two ended highly arc transitive digraphs. Although we cannot state this just yet, we shall introduce here a further generalization of the construction from the previous section which will be utilized.

Construction 4 We define a *coloured template* to be a digraph K equipped with a possibly improper colouring of the edges $\varphi : E(K) \to \{1,\ldots,t\}$ and also equipped with a distinguished partition of the vertices into sets $V_0, V_1, \ldots, V_m$ so that every edge goes from a point in V_i to a point in V_{i+1} for some $0 \leq i < m$. Given such a template K, we define the digraph $\overleftrightarrow{K}$ to have vertex set $\mathbb{Z} \times V_0 \times V_1 \ldots \times V_m$ and an edge from $(i, x_0, x_1, \ldots, x_m)$ to $(i+1, y_0, y_1, \ldots, y_m)$ whenever all of the arcs $(x_0,y_1), (x_1,y_2), \ldots, (x_{m-1},y_m)$ are present in K and all have the same colour.

Our structure theorem will show that all two ended highly arc transitive digraphs either have a type of quotient by which we can reduce them, or up to

vertex cloning, they can all be represented using the above construction. The proof of this will be built to slowly with many small lemmas. We shall work extensively with group actions in this section, and our groups shall act on the left. For clarity, we shall always use upper case greek letters for groups and lower case greek letters for elements of groups. If Ψ is a group and $\Lambda \leq \Psi$ we let Ψ/Λ denote the set of left Λ-cosets in Ψ.

Throughout this section, we shall always assume that G is a highly arc-transitive digraph so that the underlying undirected graph is connected and has two ends. For any partition $\mathcal{P}$ of the vertices, we let $G^{\mathcal{P}}$ denote the graph obtained from G by identifying the vertices in each block of $\mathcal{P}$ to a single new vertex and then deleting any parallel edges. We say that a system of imprimitivity $\mathcal{B}$ is a *$\mathbb{Z}$-system* if $G^{\mathcal{B}}$ is isomorphic to two way infinite directed path. In this case the blocks of $\mathcal{B}$ may be enumerated $\{B_i\}_{i\in\mathbb{Z}}$ so that every edge has tail in B_i and head in B_{i+1} for some $i \in \mathbb{Z}$. Note that in this case, we have that for every $\varphi \in Aut(G)$ there exists $j \in \mathbb{Z}$ so that $\varphi(B_i) = B_{i+j}$ for every $i \in \mathbb{Z}$.

Proposition 4.1 *The digraph G has a unique $\mathbb{Z}$-system $\mathcal{B}$. Furthermore, every system of imprimitivity with finite blocks is a refinement of $\mathcal{B}$ (in particular, there is a unique $\mathbb{Z}$-system).*

Note: much of this Proposition was already proved by Cameron, Praeger, and Wormald, and is included for reader's convenience.

Proof: Every connected vertex transitive two ended graph has a system of imprimitivity $\mathcal{B}$ with finite blocks and an (infinite) cyclic relation on $\mathcal{B}$ which is preserved by the automorphism group (this follows, for instance from Dunwoody's theorem [3] on cutting up graphs). Enumerate the blocks $\{B_i\}_{i\in\mathbb{Z}}$ so that this cyclic relation associates B_i with B_{i-1} and B_{i+1} for every $i \in \mathbb{Z}$. Now, it follows from the assumption that G is arc-transitive that there exists a fixed integer k so that every edge with one end in B_i and one end in B_j satisfies $|i - j| = k$. It then follows from the connectivity of the underlying graph that $k = 1$. So, every edge has its ends in two consecutive blocks of $\{B_i\}_{i\in\mathbb{Z}}$.

Suppose (for a contradiction) that there exists a directed path of length two with vertex order x_0, x_1, x_2 so that both x_0 and x_2 are in the same block of $\mathcal{B}$. In this case, it follows from the arc-transitivity of G that we may map this path to a path with vertex sequence x_1, x_2, x_3 for some x_3 which is in the same block as x_1. Continuing in this manner, we will eventually form a closed directed walk which has only finitely many vertices (as they must be contained in just two blocks of $\mathcal{B}$). Since this violates the assumption that G is highly arc-transitive, we conclude that every vertex $x \in B_i$ has the property that either $N^-(x) \subseteq B_{i-1}$ and $N^+(x) \subseteq B_{i+1}$ or the property that $N^-(x) \subseteq B_{i+1}$ and $N^+(x) \subseteq B_{i-1}$. If there are vertices of both of these types in the block B_0, then every block must have vertices of both types, but then the nonexistence of a directed path of length two as above forces the underlying graph to be disconnected. Since this

is contradictory, we may assume (by possibly reversing our ordering) that every edge has its tail in some block B_i and its head in B_{i+1}. It follows from this that $\mathcal{B}$ is a $\mathbb{Z}$-system.

For the last part of the theorem, we let $\mathcal{C}$ be a system of imprimitivity with finite blocks, and suppose (for a contradiction) that $\mathcal{C}$ is not a refinement of $\mathcal{B}$. Choose a block C of $\mathcal{C}$ and let $i \in \mathbb{Z}$ be the smallest integer with $B_i \cap C \neq \emptyset$ and let $j \in \mathbb{Z}$ be the largest integer with $B_j \cap C \neq \emptyset$ (and note that $i < j$). Now choose a vertex $u \in B_i \cap C$ and $v \in B_j \cap C$ and choose an automorphism φ so that $\varphi(u) = v$. It now follows that $\varphi(C) = C$ and that $\varphi(B_k) = B_{k+j-i}$ for every $k \in \mathbb{Z}$, but this is a contradiction. Thus, $\mathcal{C}$ must be a refinement of $\mathcal{B}$. It follows immediately from this that there is a unique $\mathbb{Z}$-system. $\square$

Lemma 4.2 *There exists a nontrivial automorphism of G with only finitely many non-fixed points.*

Proof: Let $\mathcal{B} = \{B_i\}_{i \in \mathbb{Z}}$ be the $\mathbb{Z}$-system, and suppose that every vertex has outdegree d and that each block of $\mathcal{B}$ has size k. Next, choose an integer n large enough so that $d^n > (k!)^2$ and consider a directed path P of length n with vertex sequence $x_0, x_1, \ldots, x_n$ with $x_i \in B_i$. Now, there are d^n directed paths of length n which start at the vertex x_0, and for each of them, we may choose an automorphism which maps P to this path. Since $d^n > (k!)^2$ it follows that there must be two such automorphisms, say φ_1 and φ_2 which give exactly the same permutation of both B_0 and B_n. It follows that the automorphism $\psi = \varphi_1 \varphi_2^{-1}$ is nontrivial, but gives the identity permutation on both B_0 and B_n. Now, we define a mapping $\psi' : V(G) \to V(G)$ by the following rule

$$\psi'(x) = \begin{cases} \psi(x) & \text{if } x \in B_1 \cup B_2 \cup \ldots B_{n-1} \\ x & \text{otherwise} \end{cases}$$

It is immediate that ψ' is a nontrivial automorphism which has only finitely many non-fixed points, as desired. $\square$

Based on the above lemma, there exists a smallest integer ℓ so that G has a nontrivial automorphism which fixes all but $\ell + 1$ blocks from the $\mathbb{Z}$-system pointwise. It is immediate that every such automorphism which is nontrivial must give a non-identity permutation on $\ell + 1$ consecutive blocks and the identity on all others. For every integer i, let Γ_i denote the subgroup of automorphisms which pointwise fix all blocks of the $\mathbb{Z}$-system with the (possible) exception of $B_{i-\ell}, B_{i-\ell+1}, \ldots, B_i$. We let Γ denote the subgroup of $\mathrm{Aut}(G)$ generated by $\cup_{i \in \mathbb{Z}} \Gamma_i$.

Lemma 4.3 *We have*

1. *If $\alpha \in \Gamma_i$ and $\beta \in \Gamma_j$ with $i \neq j$, then α and β commute.*

2. *If $\varphi \in \mathrm{Aut}(G)$ satisfies $\varphi(B_0) = B_k$ then $\varphi\Gamma_j\varphi^{-1} = \Gamma_{j+k}$.*

3. *$\Gamma \triangleleft \mathrm{Aut}(G)$.*

Proof: To prove 1, we consider the mapping $\gamma = \alpha\beta\alpha^{-1}\beta^{-1}$. Since α pointwise fixes all blocks but $B_{i-\ell}, B_{i-\ell+1}, \ldots, B_i$ and β pointwise fixes all blocks but $B_{j-\ell}, B_{j-\ell+1}, \ldots, B_j$ the map γ must pointwise fix any block, which is not in both of these lists. However, then γ must pointwise fix all but fewer than $\ell + 1$ blocks, so γ is the identity.

For 2, we first note that $\varphi(B_i) = B_{i+k}$ for every $i \in \mathbb{Z}$. Now, for every $\alpha \in \Gamma_j$ we have that $\varphi\alpha\varphi^{-1}$ must pointwise fix all blocks except possibly $B_{j+k-\ell}, B_{j+k-\ell+1}, \ldots, B_{j+k}$ and it follows that $\varphi\alpha\varphi^{-1} \in \Gamma_{j+k}$ which gives 2.

To prove 3, let $\alpha \in \Gamma$ and express this element as $\alpha = \alpha_1\alpha_2 \ldots \alpha_m$ where each α_i is in a subgroup of the form Γ_j. Now we have

$$\varphi\alpha\varphi^{-1} = (\varphi\alpha_1\varphi^{-1})(\varphi\alpha_2\varphi^{-1}) \ldots (\varphi\alpha_m\varphi^{-1})$$

so $\varphi\alpha\varphi^{-1}$ is also contained in Γ. □

We call a two way infinite directed path a *line*. The following lemma may be proved with a straightforward compactness argument.

Lemma 4.4 *Let $\mathbf{x}, \mathbf{y}$ be lines in G with x a point in $\mathbf{x}$ and y a point in $\mathbf{y}$. Then there exists an automorphism φ of G which maps $\mathbf{x}$ to $\mathbf{y}$ and maps x to y.*

Lemma 4.5 *Let $\Lambda \triangleleft \mathrm{Aut}(G)$ and let $\mathcal{C}$ be the partition of V given by the orbits under the action of Λ.*

1. *$\mathcal{C}$ is a system of imprimitivity.*

2. *If $C, C' \in \mathcal{C}$ and there is an edge from C to C' then every point in C has an outneighbour in C' and every point in C' has an inneighbour in C.*

3. *$G^{\mathcal{C}}$ is highly arc transitive.*

4. *If $\mathbf{x}$ is a line, then the graph $G_{\mathbf{x}}$ induced by the union of those blocks of $\mathcal{C}$ which contain a point in $\mathbf{x}$ is highly arc transitive.*

5. *The graphs $G_{\mathbf{x}}$ (for a line $\mathbf{x}$) are all isomorphic.*

Proof: Part 1 is a standard fact about group actions. For the proof, let $u, v \in V$ be in the same orbit of Λ, say $u = \alpha(v)$ for $\alpha \in \Lambda$, and let φ be any automorphism. Now $\varphi(u) = \varphi\alpha(v) = \varphi\alpha\varphi^{-1}\varphi(v)$ so $\varphi(u)$ and $\varphi(v)$ are also in the same orbit of Λ.

For part 2, choose an edge $(u, u') \in E$ with $u \in C$ and $u' \in C'$. Now, for every $v \in C$ there is an element in Λ which maps u to v. Since this element must fix C' setwise, it follows that v has a outneighbour in C'. A similar argument shows that every point in C' has an inneighbour in C.

Now, to prove 3, we let $C_1, C_2, \ldots, C_k$ and $C'_1, C'_2, \ldots, C'_k$ be two sequences of blocks of $\mathcal{C}$ so that both form the vertex set of a directed path in the graph $G^{\mathcal{C}}$. Using 2 we may choose vertex sequences $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_k$ in G so that $x_i \in C_i$ and $x'_i \in C'_i$ for $1 \leq i \leq k$ and so that $(x_i, x_{i+1}), (x'_i, x'_{i+1}) \in E$ for $1 \leq i \leq k-1$. It follows from the high arc transitivity of G that there is an automorphism φ of G so that $\varphi(x_i) = x'_i$ for $1 \leq i \leq k$. Then $\varphi(C_i) = C'_i$ for $1 \leq i \leq k$ so φ gives an automorphism of $G^{\mathcal{C}}$ which maps $C_1, \ldots, C_k$ to $C'_1, \ldots, C'_k$. It follows that $G^{\mathcal{C}}$ is highly arc transitive.

For the proof of 4, set X to be he union of those blocks of $\mathcal{C}$ which contain a point of $\mathbf{x}$, and set G' to be the graph induced by X. Now we let $y_1, \ldots y_k$ and $y'_1, \ldots, y'_k$ be vertex sequences of paths in G'. It follows from 2 that we may extend $y_1, \ldots y_k$ and $y'_1, \ldots, y'_k$ respectively to lines $\mathbf{y}$ and $\mathbf{y}'$ in G'. It now follows from the previous lemma that there is an automorphism φ of G which maps $\mathbf{y}$ to $\mathbf{y}'$ and further has $\varphi(y_i) = y'_i$ for $1 \leq i \leq k$. It then follows that $\varphi(X) = X$ so φ yields an automorphism of G' which sends $y_1, \ldots, y_k$ to $y'_1, \ldots, y'_k$. We conclude that G' is highly arc transitive.

Part 5 follows easily from Lemma 4.4. $\square$

We define G to be *essentially primitive* if there does not exist $\Lambda \triangleleft \mathrm{Aut}(G)$ so that the orbits of Λ on V generate a proper nontrivial system of imprimitivity which is not equal to the $\mathbb{Z}$-system. Parts 3–5 from the previous lemma show that any two ended highly arc transitive digraph which is not essentially primitive has a type of decomposition into a highly arc transitive subgraph and a highly arc transitive quotient. Although this decomposition does not seem to give us a construction, we will focus the remainder of this section on understanding the structure of the essentially primitive digraphs. Note, however, that we do not know whether this is truly needed: the only examples of highly arc transitive digraphs that are not essetially primitive that we are aware of are a disjoint union of two highly arc transitive digraphs (rather trivial example) and graphs obtained by a *horocyclic product* − however, such product of two highly arc transitive digraphs obtained by our template construction can also be obtained by such construction.

Lemma 4.6 *If G is essentially primitive, then the orbits under the action of Γ are $\{B_i : i \in \mathbb{Z}\}$.*

Proof: This follows immediately from Lemma 4.3. $\square$

Next we shall introduce another useful subgroup of $\mathrm{Aut}(G)$. Choose an automorphism τ so that $\tau(B_0) = B_1$ (so, more generally, $\tau(B_i) = B_{i+1}$), and let

Φ be the subgroup of $\mathrm{Aut}(G)$ which is generated by τ and Γ. We will use Φ to describe our graph, so let us record some key features of it.

Lemma 4.7

1. *$\tau^{-1}\Gamma_k\tau = \Gamma_{k-1}$*
2. *$\Gamma \lhd \Phi$*
3. *$\langle\tau\rangle \cong \mathbb{Z}$*
4. *$\Gamma \cap \langle\tau\rangle = \{1\}$*
5. *Φ is a semidirect product of $\langle\tau\rangle$ and Γ.*

Proof: Immediate from Lemma 4.3. $\square$

Next we introduce another family of useful subgroups of Φ. For every $j \leq k$ we define $\overline{\Gamma}_{j..k}$ to be the subgroup of Γ generated by $\left(\bigcup_{i<j}\Gamma_i\right) \cup \left(\bigcup_{i>k}\Gamma_i\right)$. Note that $\overline{\Gamma}_{0..\ell}$ is precisely the subgroup of Γ consisting of those automorphisms which act trivially on B_0.

Lemma 4.8

1. *Every $\overline{\Gamma}_{j..k}$ coset in Φ has a unique representation as $\tau^m\left(\prod_{i=j}^{k}\alpha_i\right)\overline{\Gamma}_{j..k}$ where $\alpha_i \in \Gamma_i$ for every $j \leq i \leq k$ (henceforth we call this the* standard *form).*
2. *$\tau^{-1}\bar{\Gamma}_{j..k}\tau = \bar{\Gamma}_{j-1..k-1}$*
3. *If $A \subseteq \tau\Gamma$ then $\bar{\Gamma}_{j..k}A = A\bar{\Gamma}_{j-1..k-1}$.*
4. *A set $A \subseteq \tau\Gamma$ satisfies $\bar{\Gamma}_{j..k}A\bar{\Gamma}_{j..k} = A$ if and only if $A\bar{\Gamma}_{j..k-1} = A$.*

Proof: The first and second properties follows immediately from the previous lemma. For the third, choose $A' \subseteq \Gamma$ so that $A = \tau A'$ and observe that

$$\bar{\Gamma}_{j..k}A = \bar{\Gamma}_{j..k}\tau A' = \tau\bar{\Gamma}_{j-1..k-1}A' = \tau A'\bar{\Gamma}_{j-1..k-1} = A\bar{\Gamma}_{j-1..k-1}.$$

To prove the last property it is enough to observe that for $A \subseteq \tau\Gamma$

$$\bar{\Gamma}_{j..k}A\bar{\Gamma}_{j..k} = A\bar{\Gamma}_{j-1..k-1}\bar{\Gamma}_{j..k} = A\bar{\Gamma}_{j..k-1}\,.$$

$\square$

The only additional ingredients required for our structure theorem are some standard properties of vertex transitive graphs. Let Ψ be a group, let $\Lambda \leq$

Ψ, and let $A \subseteq \Psi$ satisfy $\Lambda A \Lambda = A$. Then we define the *Cayley coset graph* $\mathrm{Cayley}(\Psi/\Lambda, A)$ to be the graph with vertex set Ψ/Λ where there is an edge from Q to R if $Q^{-1}R \subseteq A$. The group Ψ has a natural action on the vertices by left multiplication, and this action preserves the edges, and is transitive. The following essential (but standard) proposition shows that every vertex transitive graph is isomorphic to a Cayley coset graph. Here, if Ψ acts on a set X and $u \in X$ we let $\Psi_u = \{\gamma \in \Psi : \gamma(u) = u\}$ denote the point stabilizer of u.

Proposition 4.9 *Let H be a graph, let $u \in V(H)$ and let $\Phi \leq \mathrm{Aut}(H)$ act transitively on $V(H)$. Then there exists $A \subseteq \Phi$ so that $H \cong \mathrm{Cayley}(\Phi/\Phi_u, A)$, and further this isomorphism may be chosen to map u to Φ_u.*

Proof: (sketch – included for the reader's convenience) Associate each $v \in V(H)$ with the set $\{\gamma \in \Phi : \gamma(u) = v\}$ (this is the standard realization of the action of Φ on $V(H)$ with base point u). Now, define A to be the union of those Φ_u cosets associated with a point v with $(u, v) \in E(H)$. It follows immediately from this construction that $A\Phi_u = A$ and since Φ_u fixes u we must further have $\Phi_u A = A$. It is also immediate that $(u, v) \in E(H)$ if and only if v is associated with a Φ_u-coset contained in A. But then, for every $u', v' \in V(H)$, if u' is associated with $\alpha\Phi_u$ and v' is associated with $\beta\Phi_u$, the transformation α^{-1} maps u' to u and v' to the vertex associated with $\alpha^{-1}\beta\Phi_u$ (preserving adjacency). So, $(u', v') \in E(H)$ if and only if $\alpha^{-1}\beta\Phi_u \subseteq A$, which is equivalent to $(\alpha\Phi_u)^{-1}\beta\Phi_u \subseteq A$ as desired. $\square$

Proposition 4.10 *Let $G = Cayley(\Phi/\Lambda, A)$ and let $\Lambda' \leq \Lambda$ with $[\Lambda : \Lambda'] = k$. Then $G' = Cayley(\Phi/\Lambda', A)$ is a Cayley coset graph which is isomorphic to the graph obtained from G by cloning each vertex $k - 1$ times.*

Proof: (sketch) By definition, in the graph G' there will be an edge from $Q \in \Phi/\Lambda'$ to $R \in \Phi/\Lambda$ if $Q^{-1}R \subseteq A$. If R and R' lie in the same Λ-coset then $Q^{-1}R\Lambda = Q^{-1}R'\Lambda$. Since $A\Lambda = A$ it follows that there is an edge from Q to R if and only if there is an edge from Q to R'. So, two vertices which lie in the same Λ-coset will have the same inneighbours. A similar argument shows that they have the same outneighbours. Thus, G' is isomorphic to the graph obtained from G by cloning each vertex exactly $k - 1$ times. $\square$

Theorem 4.11 *If $G = (V, E)$ is essentially primitive, then there exists a digraph G^+ obtained from G by cloning each vertex the same (finite) number of times and a coloured template K so that $G^+ \cong \overleftrightarrow{K}$.*

Proof: It follows immediately from Lemma 4.6 that the group Φ acts transitively on V. Choose a vertex $u \in B_0$ and apply Proposition 4.9 to choose $A \subseteq \Phi$ so that

$G \cong \mathrm{Cayley}(\Phi/\Phi_u, A)$. Since Φ_u is the stabilizer of u and $\overline{\Gamma}_{0..\ell}$ is the subgroup of Φ which fixes every point in B_0 we must have $\overline{\Gamma}_{0..\ell} \leq \Phi_u \leq \Phi$ (and note that this also implies that $[\Phi_u : \overline{\Gamma}_{0..\ell}]$ is finite). It now follows from Proposition 4.10 that $G^+ = \mathrm{Cayley}(\Phi/\overline{\Gamma}_{0..\ell}, A)$ is obtained from G by cloning each vertex the same number of times, so it shall suffice to prove that G^+ can be obtained from our construction.

By assumption, A must satisfy $\overline{\Gamma}_{0..\ell} A \overline{\Gamma}_{0..\ell} = A$ and then it follows from Lemma 4.8 that $A\overline{\Gamma}_{0..\ell-1} = A$ so we may partition A into $\overline{\Gamma}_{0..\ell-1}$ cosets as $\{A_1, A_2, \ldots, A_t\}$. Now, each A_q also satisfies $\overline{\Gamma}_{0..\ell} A_q \overline{\Gamma}_{0..\ell} = A_q$, so we may define a Cayley graph $G_q^+ = \mathrm{Cayley}(\Phi/\overline{\Gamma}_{0..\ell}, A_q)$ and now G^+ is the edge-disjoint union of the graphs $G_1^+, \ldots, G_t^+$. We may now view each $q = 1, \ldots, t$ as a colour and view G^+ as having its edges coloured accordingly.

Fix $1 \leq q \leq t$ and consider the graph G_q^+ and let $A_q = \tau \left(\prod_{i=0}^{\ell-1} \gamma_i \right) \overline{\Gamma}_{0..\ell-1}$ in standard form. Let $v = \tau^k \left(\prod_{i=0}^{\ell} \alpha_i \right) \overline{\Gamma}_{0..\ell}$ be a vertex of G_q^+ in standard form. Within the graph G_q^+, the vertex v will have outneighbours consisting of exactly those $\overline{\Gamma}_{0..\ell}$ cosets contained in the set

$$\begin{aligned}
vA_q &= \tau^k \left(\prod_{i=0}^{\ell} \alpha_i \right) \overline{\Gamma}_{0..\ell}\, \tau \left(\prod_{i=0}^{\ell-1} \gamma_i \right) \overline{\Gamma}_{0..\ell-1} \\
&= \tau^{k+1} \left(\prod_{i=0}^{\ell} \tau^{-1}\alpha_i\tau \right) \tau^{-1}\overline{\Gamma}_{0..\ell}\, \tau \left(\prod_{i=0}^{\ell-1} \gamma_i \right) \overline{\Gamma}_{0..\ell-1} \\
&= \tau^{k+1} \left(\prod_{i=1}^{\ell} \tau^{-1}\alpha_i\tau \right) \left(\prod_{i=0}^{\ell-1} \gamma_i \right) \overline{\Gamma}_{0..\ell-1} \\
&= \tau^{k+1} \left(\prod_{i=1}^{\ell} \tau^{-1}\alpha_i\tau\, \gamma_{i-1} \right) \overline{\Gamma}_{0..\ell-1}
\end{aligned}$$

In other words, a vertex w is an outneighbour of v if and only if in standard form $w = \tau^{k+1} \left(\prod_{i=0}^{\ell} \beta_i \right) \overline{\Gamma}_{0..\ell}$ where $\beta_{i-1} = \tau^{-1}\alpha_i\tau\gamma_{i-1}$ for every $1 \leq i \leq \ell$ (and there is no restriction on β_ℓ). Next we shall define a template K_q with vertex tuple $(\Gamma_\ell, \Gamma_{\ell-1}, \ldots, \Gamma_0)$ and an edge from $\delta \in \Gamma_i$ to $\epsilon \in \Gamma_{i-1}$ if and only if $\epsilon = \tau^{-1}\delta\tau\gamma_{i-1}$. It now follows that (v, w) is an edge of G_q^+ if and only if (using standard form) $v = \tau^i \alpha_0\alpha_1 \ldots \alpha_\ell \overline{\Gamma}_{0..\ell}$ and $w = \tau^j \beta_0\beta_1 \ldots \beta_\ell \overline{\Gamma}_{0..\ell}$ satisfy $j = i+1$ and (α_i, β_{i-1}) is an edge of K_q for every $1 \leq i \leq \ell$. It follows from this that $G_q^+ \cong \overleftrightarrow{K}_q$ by way of the isomorphism which maps a vertex $v = \tau^i \alpha_0\alpha_1 \ldots \alpha_\ell \overline{\Gamma}_{0..\ell}$ of G_q^+ to the vertex $(i, \alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_0)$ of $\overleftrightarrow{K}_q$.

We now define K to be a coloured template with vertex set $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_\ell$, vertex partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell\}$, and an edge from $\delta \in \Gamma_i$ to $\epsilon \in \Gamma_{i+1}$ of colour q if and only if this edge exists in the template K_q. It now follows that $G^+ \cong \overleftrightarrow{K}$ which completes the proof. $\square$

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